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Geometry of Crystal Structure with Defects. II. Non-Euclidean Picture

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The following point of view is geometrically formulated and its consequences examined: the lattice of a crystalline body with a continuous distribution of dislocations can be locally described as an ideal lattice in non-Euclidean space. The types of distribution of dislocations are described by the classification of three-dimensional real Lie algebras. The influence of point defects and the elastic deformation field on the geometry of the material structure of a crystalline body with dislocations is examined. The case where a crystal with dislocations reacts as a body with internal rotational degrees of freedom is discussed.

1. Introduction

In Part I of this work (Trzęsowski, 1987) it is shown that the description of distorted crystal structure can be based on a consideration of a distribution of lattice groups $T(P)$, $P \in \mathfrak{B}$, described by the nonintegrable distribution of lattice bases $E_{T(P)}$, $P \in \mathfrak{B}$:

$$E_{T(P)} = (\underline{E}_a(P);\ a = 1, 2, 3), \qquad \underline{E}_a(P) \in T_p(\mathfrak{B}) \tag{1}$$

where $\mathfrak{B}$ denotes a body treated as a three-dimensional, smooth, and connected differentiable manifold (Part I, Section 3) and $T_P(\mathfrak{B})$ is the space tangent to $\mathfrak{B}$ in $P \in \mathfrak{B}$. This is equivalent to describing a body material structure with defects by introducing a certain teleparallelism Φ on the body, i.e., the system of linear isomorphisms [Part I, formulas (75)-(77)]

$$\Phi = \{\Phi_{PQ}\colon T_P(\mathfrak{B}) \to T_Q(\mathfrak{B}),\ P, Q \in \mathfrak{B}\}$$

of tangent spaces, with the consistency condition

$$\Phi_{QP} \circ \Phi_{PR} = \Phi_{QR}, \qquad \Phi_{PP} = \text{ident} \cdot (P, Q, R \in \mathfrak{B})$$

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and with the regularity condition: if $\underset{\sim}{v}_P \in T_P(\mathfrak{B})$ is a fixed vector and

$$\underset{\sim}{v} = \{\underset{\sim}{v}_Q, Q \in \mathfrak{B}; \underset{\sim}{v}_Q = \Phi_{PQ}\underset{\sim}{v}_P, \Phi_{PQ} \in \Phi\}$$

then $\underset{\sim}{v}$ is a smooth vector field, called a Φ-parallel vector field.

We also endow the body with three Riemannian metric tensors: the right Cauchy-Green tensor $\underset{\sim}{C}$ [Part I, formula (71)] induced on the body by the deformation of its solid figure, the Φ-parallel tensor $\underset{\sim}{g}$ [Part I, formula (73)] describing the distortion of the metric structure of the lattice [Part I, commentary after formula (34)], and the tensor $\underset{\sim}{G}$ defined by the elastic distortion [Part I, formulas (86)-(88)]. So, there appears the problem of unifying this geometrical description. Tensors $\underset{\sim}{C}$ and $\underset{\sim}{G}$ have the character of field variables and that is why they are not useful in constructing the "geometric skeleton" of the theory. But the teleparallelism Φ and the Φ-parallel metric tensor $\underset{\sim}{g}$ have the character of absolute objects, and that is why the analysis of the relations between them is of basic importance. In this paper it is proposed to interconnect these two geometric objects by an appropriate generalization of the notion of the lattice line (of the Bravais lattice; see Part I, Section 2). The basic consequence of such an approach to the description of the crystal structure distortion is examined.

2. STRUCTURALLY UNIFORM CRYSTALLINE BODY

Let Φ be a teleparallelism defined on the body $\mathfrak{B}$ (see Section 1). It is known (e.g., Sikorski, 1972; Wolf, 1972) that for the connected differentiable manifold $\mathfrak{B}$ there exists a natural one-one correspondence between teleparallelisms Φ and smooth trivializations

$$E[\Phi] = (\underset{\sim}{E}_1, \underset{\sim}{E}_2, \underset{\sim}{E}_3): \quad \mathfrak{B} \to B(\mathfrak{B}) \tag{2}$$

of the frame bundle $B(\mathfrak{B})$. If $\mathfrak{B}$ is additionally simply connected, then there also exists a one-one correspondence between teleparallelisms Φ on $\mathfrak{B}$ and flat (i.e., with vanishing curvature tensor) linear connections $\Lambda[\Phi]$ on $\mathfrak{B}$. Because of this we will assume in this paper that the body $\mathfrak{B}$ is additionally simply connected.

If $E[\phi]$ is a trivialization (2) of the teleparallelism Φ, $X = (X^A)$ a coordinate system on $\mathfrak{B}$, and

$$\underset{\sim}{E}_a(X(P)) = \underset{a}{e}^A(X(P))\,\partial_A \in T_P(\mathfrak{B})$$

$$E^a(X(P)) = \overset{a}{e}_A(X(P))\,dX^A \in T_P^*(\mathfrak{B}) \tag{3}$$

$$\underset{a}{e}^A(X)\overset{b}{e}_A(X) = \delta^b_a, \qquad \overset{a}{e}_A(X)\underset{a}{e}^B(X) = \delta^B_A$$

then the teleparallelism connection $\Lambda[\phi]$ has the following form (Yano, 1955):

$$\lambda^A{}_B[\Phi] = -\overset{a}{e}_B\, d\underset{a}{e}^A = \underset{a}{e}^A\, d\overset{a}{e}_B = \Lambda^A_{BC}\, dX^C$$
$$\Lambda^A{}_{BC} = \Lambda_{BC}{}^A = \underset{a}{e}^A\, \partial_B \overset{a}{e}_C \tag{4}$$

The curvature tensor of the connection $\Lambda[\Phi]$ vanishes and its torsion tensor has the following form:

$$\underset{\sim}{S}[\Phi] = -\underset{\sim}{E}_a \otimes dE^a = S_{BC}{}^A\, dX^B \otimes dX^C \otimes \partial_A$$
$$S_{BC}{}^A = \Lambda_{[BC]}{}^A = \tfrac{1}{2}\underset{a}{e}^A(\partial_C \overset{a}{e}_B - \partial_B \overset{a}{e}_C) \tag{5}$$

so that

$$\underset{\sim}{S}[\Phi] = 0 \Leftrightarrow E^a = d\overset{a}{e} \tag{6}$$

An arbitrary Φ-parallel metric tensor $\underset{\sim}{g}$ has the form

$$\underset{\sim}{g}(X) = g_{ab} E^a(X) \otimes E^b(X)$$
$$= g_{AB}(X)\, dX^A \otimes dX^B \tag{7}$$
$$g_{AB}(X) = \overset{a}{e}_A(X) \overset{b}{e}_B(X) g_{ab}, \qquad g_{ab} = \text{const}$$

The vanishing of the torsion tensor $\underset{\sim}{S}[\Phi]$ implies the vanishing of the curvature tensor $\underset{\sim}{R}[g]$ of the Levi-Civita connection for $\underset{\sim}{g}$. But the inverse implication is not true. It follows from the fact that the case

$$R_{ABC}{}^D[g] = 0 \tag{8}$$

[Part I, (62)] permits the existence of a nonintegrable moving frame (3) such that

$$\underset{a}{e}^A(X) = \underset{b}{S}^A Q^b_a(X)$$
$$\underset{\sim}{Q}(X) = \|Q^a_b(X)\| \in O_g(R^3), \qquad \|\underset{a}{S}^A\| \in GL(R^3) \tag{9}$$

where $O_g(R^3)$ denotes the group of g-orthogonal matrices [Part I, (8)]; in this case

$$g_{AB}(X) = \delta_{AB} \tag{10}$$

The volume form defined by tensor $\underline{g}$ has the following form:

$$\begin{aligned}\omega(X) &= V_T E^1(X) \wedge E^2(X) \wedge E^3(X) \\ &= V_e(X)\, dX^1 \wedge dX^2 \wedge dX^3 \\ V_e(X) &= [\det\|g_{AB}(x)\|]^{1/2} = V_T e(X) \\ e(X) &= |\det\| \overset{a}{e}_A(X)\| \,|, \qquad V_T = (\det\|g_{ab}\|)^{1/2}\end{aligned} \tag{11}$$

where V_T is the volume of the undistorted lattice primitive cell [Part I, (3)-(7) and (33)].

Let us denote by ∇^λ the covariant derivative for the teleparallelism connection $\Lambda[\Phi]$. Then [cf. (4) and (11)]

$$\overset{\lambda}{\nabla} \underline{E}_a = 0, \qquad \overset{\lambda}{\nabla} E^a = 0 \tag{12}$$

and

$$\overset{\lambda}{\nabla} \underline{g} = 0, \qquad \overset{\lambda}{\nabla} \omega = 0 \tag{13}$$

Equations (12) are a non-Euclidean generalization of the Euclidean constancy of the vector basis of the ideal (i.e., without defects) lattice. Equations (13) are a consequence of the postulate of metric uniformity [Part I, condition (33)] and they mean that the metric structure of distorted lattice is represented, in the space with teleparallelism, by the metric structure of the nondistorted lattice.

The conditions (12) and (13) do not ensure the reconstruction of all basic properties of the Euclidean Bravais lattice. For example, the property of this lattice that the lattice lines (Part I, Section 2) are geodesics of Euclidean parallelism has been lost. It follows from the fact that the first relations of (13) is not enough to generalize the known theorem that each translation-invariant metric on R^n is consistent with Euclidean parallelism. The generalization of this theorem on the space with teleparallelism was formulated by Wold (1972). Namely, teleparallelism Φ and the metric g (pseudo-Riemannian or Riemannian) on $\mathfrak{B}$ are called *consistent* if:

(i) g is Φ-invariant, i.e.,

$$\forall P, Q \in \mathfrak{B} \quad \forall \underline{v}, \underline{w} \in T_P(\mathfrak{B}) \qquad g_P(\underline{v}, \underline{w}) = g_Q(\Phi_{QP}\underline{v}, \Phi_{QP}\underline{w})$$

(ii) g-geodesics are, modulo parametrization, Φ-geodesics.

Let us quote, after Wolf, two theorems describing the basic properties of the given understood consistency of Φ and g.

Theorem 1. If g is a Φ-invariant metric (pseudo-Riemannian or Riemannian) on the manifold $\mathfrak{B}$, then the following conditions are equivalent:

1. Φ and g are consistent.

2. If Γ^A_{BC} is a Levi-Civita connection for g, Λ^A_{BC} a teleparallelism connection for Φ, and $S_{AB}{}^C$ the torsion tensor for this teleparallelism connection, then

$$\Gamma^A_{BC} = \Lambda^A_{BC} - S_{BC}{}^A \tag{14}$$

Theorem 2. Let $(\mathfrak{B}, g)$ be a connected pseudo-Riemannian or Riemannian manifold. $\mathfrak{B}$ has teleparallelism Φ consistent with g if and only if there exists on $\mathfrak{B}$ a global moving frame $[\underline{E}_a(P), P \in \mathfrak{B};\ a = 1, 2, 3]$ such that:

1. Each $\underline{E}_a$ is a Killing vector field on $(\mathfrak{B}, g)$, i.e. [in the designations (3)],

$$\overset{g}{\nabla}_A \underset{a}{e}_B + \overset{g}{\nabla}_B \underset{a}{e}_A = 0, \qquad \underset{a}{e}_A = g_{AB} \underset{a}{e}^B \tag{15}$$

where ∇^g denotes covariant derivative for the connection Γ^A_{BC}.

2. g has constant components in the base $\underline{E}_a$:

$$\forall P \in \mathfrak{B} \qquad g_P(\underline{E}_a(P), \underline{E}_b(P)) = g_{ab} = \text{const} \tag{16}$$

Then $\underline{E}_a$ are Φ-parallel fields.

If $\underline{S}[\Phi]$ is the torsion tensor defined by (5), then (Yano, 1955; Wolf, 1972)

$$[\underline{E}_a, \underline{E}_b] \times \gamma^c_{ab} \underline{E}_c \tag{17}$$

$$\gamma^c_{ab} = -2S_{ab}{}^c, \qquad S_{ab}{}^c = S_{AB}{}^C \overset{c}{e}_C \underset{a}{e}^A \underset{b}{e}^B$$

Hence (Wolf, 1972)

$$dE^a = S_{bc}{}^a E^b \wedge E^c \tag{18}$$

and, from condition (14),

$$\Gamma^a_{bc} = -S_{bc}{}^a \tag{19}$$

So, if the teleparallelism Φ and the Φ-parallel metric g are consistent, then there is defined on $\mathfrak{B}$ an affine connection of the form:

$$\begin{aligned} \omega^a &= \Gamma^a_A\, dX^A, \qquad \Gamma^a_A = \overset{a}{e}_A \\ \omega^a_b &= \Gamma^a_{bc} \omega^c, \qquad \Gamma^a_{bc} = -S_{bc}{}^a \end{aligned} \tag{20}$$

The formulas (18)–(20) cover the formulas (93), (95), and (96) considered in Part I (Trzęsowski, 1987). They mean that a consistent pair (Φ, g)

describes the distribution of dislocations in the body. Namely [see Part I, commentary after formula (96)], the nonintegrability of forms ω^a means the existence of dislocations in the body (i.e., linear defects of translation type), whereas forms ω^a_b describe the type of distribution of these dislocations (see Section 3). For simplicity, we will also call such defined types of continuous distributed) dislocations. This should not be a source of misconception, because our considerations do not concern single dislocations. In the literature, the density of the distribution of dislocations is identified with the tensor α^{AB} of the form (e.g., Kröner, 1960; Minagava, 1979)

$$\alpha^{AB} = e^{ACD} S_{DC}{}^{B} \tag{21}$$

where e^{ABC} is a basic trivector of the Riemannian manifold $(\mathfrak{B}, g)$ [Part I, formula (65) with the change of $\underset{\sim}{C}$ for the metric tensor $\underset{\sim}{g}$].

Let us observe that conditions (15) and (16) mean that the vector fields $\underset{\sim}{E}_a$, $a = 1, 2, 3$, that define the teleparallelism Φ are so-called *g-translations*, that is, vector fields whose trajectories are g-geodesics (Yano, 1955). Consistency of Φ and g means that those g-geodesics are at the same time, with the same parametrization, Φ-geodesics (Wolf, 1972). So, we can observe that if we want to reconstruct in a space with teleparallelism the consistency between metric structure and the system of lattice lines appearing in the case of an ideal Euclidean Bravais lattice, we have to limit our considerations to the following class of material bodies:

Definition. The connected material body $\mathfrak{B}$ will be called a *structurally uniform* crystalline body if its material structure is described by a certain teleparallelism Φ and a Riemannian metric g consistent with it.

The global moving frame $E[\phi] = (\underset{\sim}{E}_a)$ on $\mathfrak{B}$ satisfying conditions 1 and 2 from Theorem 2 will be called the *Bravais (moving) frame* of the structurally uniform body. Trajectories of the vector fields of the Bravais frame will be called *metric lattice lines.* From the properties of infinitesimal motions in Riemannian space (Yano, 1955) it follows that two different vector fields of the Bravais frame cannot have the same metric lattice lines.

From the condition (14) it follows that the curvature tensor of the Levi-Civita connection for a metric tensor $\underset{\sim}{g}$ consistent with the teleparallelism Φ does not depend on the choice of $\underset{\sim}{g}$ and vanishes together with the vanishing of the torsion tensor $\underset{\sim}{S}[\Phi]$. Because of this, the curvature tensor will be called the *adjoint curvature tensor* of the structurally uniform body and will be denoted by $\underset{\sim}{R}[\Phi]$. It can be shown that (Yano, 1955)

$$\overset{g}{\nabla} \underset{\sim}{R}[\Phi] = \underset{\sim}{0} \tag{22}$$

with

$$R[\Phi]_{ABC}{}^{D} = S_{AB}{}^{E} S_{CE}{}^{D} \tag{23}$$

and (Wolf, 1972)

$$\overset{\lambda}{\nabla}\underset{\sim}{R}[\phi]=\underset{\sim}{0} \tag{24}$$

Hence and from (17) it follows that in the Bravais frame (2) on $\mathfrak{B}$

$$R[\Phi]_{abc}{}^{d}=-\tfrac{1}{4}\gamma^{p}_{ab}\gamma^{d}_{pc}=\text{const} \tag{25}$$

The Ricci tensor $\underset{\sim}{R}[\Phi]_{AB}$

$$R[\Phi]_{AB}=R[\Phi]_{CAB}{}^{C} \tag{26}$$

is also Φ-parallel, with

$$R[\Phi]_{ab}=R[\Phi]_{cab}{}^{c}=-\tfrac{1}{4}\gamma^{c}_{ap}|^{p}_{bc}=\text{const} \tag{27}$$

3. CLOSED TELEPARALLELISM

Let $\mathfrak{B}$ be a structurally uniform body and (Φ, g) a corresponding consistent pair. Let us denote by $\mathfrak{G}[\Phi]$ the linear space of all Φ-parallel vector fields on $\mathfrak{B}$. Then dim $\mathfrak{G}[\Phi]=3$ and fields $\underset{\sim}{v}\in\mathfrak{G}[\Phi]$ are g-translations (Wolf, 1972). Let us denote by $G[\Phi]$ the set of all (local) one-parameter groups $\varphi=\{\varphi_t\}$ generated by vector fields $\underset{\sim}{v}\in\mathfrak{G}[\Phi]$. From equation (17) it follows that we can introduce in $G[\Phi]$ the structure of the local Lie group generated by the vector fields of the Bravais frame (2) if and only if

$$\gamma^{c}_{ab}=\text{const} \tag{28}$$

or, equivalently,

$$\overset{\lambda}{\nabla}\underset{\sim}{S}[\Phi]=\underset{\sim}{0} \tag{29}$$

In the case (28), $\mathfrak{G}[\Phi]$ is a Lie algebra of $G[\Phi]$ and $G[\Phi]$ acts on $\mathfrak{B}$ simply transitively (Yano, 1955). The simple transitivity of action of $G[\Phi]$ on $\mathfrak{B}$ means that each two sufficiently close points $P, Q\in\mathfrak{B}$ can be joined by the trajectory generated by a vector field $\underset{\sim}{v}\in\mathfrak{G}[\Phi]$ with this field defined uniquely. The teleparallelism Φ satisfying the condition (29) is called the *closed teleparallelism* (Sławianowski, 1985). If, for example, $\mathfrak{B}$ is connected and there exists a flat metric tensor g consistent with Φ, then Φ is a closed teleparallelism; but from the condition (29) there does not follow the flatness of g (Wolf, 1972).

The case of a structurally uniform body with closed teleparallelism is important, because the property of the transitivity of the Bravais lattice group action (Part I, Section 2) is then additionally reconstructed. But even in this case, this reconstruction is limited to a sufficiently small neighborhood of each point of the body (simply transitive group $G[\Phi]$).

We can observe (see Section 2) that a transition from the ideal crystal to the crystal with (continuously distributed) dislocations can be described as a transition from its lattice basis consisting of Euclidean translation to the Bravais frame in the form of infinitesimal g-translations. If the teleparallelism Φ is closed, then the scope of this generalization is easy to examine, because in this came the vector field $\underset{\sim}{E}_a$, $a = 1, 2, 3$, span a three-dimensional real Lie algebra $\mathfrak{G}[\Phi]$ of Φ-parallel vector fields. Consequently, the problem is reduced to presenting the full classification of such Lie algebras. This classification is known (Barut and Rączka, 1977).

The case of Abelian Lie algebra

$$\gamma^a_{bc} = 0 \tag{30}$$

is equivalent to the vanishing of the torsion tensor $\underset{\sim}{S}[\Phi]$ of the material structure [equations (5) and (17)]. In this case the distortion is removable, i.e., the Bravais frame can, with a suitable deformation of the body, be transformed into the base of a certain ideal Bravais lattice [see (6) and Part I, (21)] and vectors $\underset{\sim}{E}_a$ become ordinary translations.

If the Lie algebra $\mathfrak{G}[\phi]$ is simple, then the symmetric tensor γ_{ab} of the form (Yano, 1955; Sławianowski, 1985)

$$\begin{aligned} \gamma_{ab} &= \gamma^c_{ad}\gamma^d_{bc} = \gamma_{AB}\underset{a}{e}^A\underset{b}{e}^B \\ -\tfrac{1}{4}\gamma_{AB} &= 4S_{AD}{}^C S_{BC}{}^D = R[\Phi]_{AB} \end{aligned} \tag{31}$$

where $\underset{\sim}{R}[\Phi]_{AB}$ is the Ricci tensor corresponding to the adjoint curvature tensor [formulas (26) and (27)], is nonsingular,

$$\det\|\gamma_{ab}\| \neq 0 \tag{32}$$

and defines a Killing metric of the Lie algebra $\mathfrak{G}[\phi]$. There are only two types of simple three-dimensional Lie algebras, represented by Lie algebra $so(3)$ of the Euclidean rotation group $SO(3)$ [tensor γ_{ab} of signature $(---)$] and Lie algebra $so(2, 1)$ of the three-dimensional Lorentz rotation group $SO(2.1)$ [tensor γ_{ab} of signature $(++-)$]. The first case describes dislocations of rotation type. The second case describes dislocations of the Lorentz rotation type, i.e., dislocations of shear type [Part I, Appendix, (A7)-(A9)]. The first to point out the possibility of describing the continuous distribution of dislocations on the plane by a two-dimensioal Minkowski metric was Żórawski (1965).

A nilpotent Lie algebra describes the dislocations of nonrotational type. For example, the Lie algebra of the Weyl group [Part I, Appendix, (A5), and formula (28)] is such an algebra. This group describes simple shear, i.e., deformations changing a cube into a parallelepiped.

Let us consider the case when $\mathfrak{G}[\Phi]$ is $e(2)$ or $e(1, 1)$ type. These are algebras of isometry groups of two-dimensional space, Euclidean [group $E(2)$] or pseudo-Euclidean [group $E(1, 1)$], respectively, and are examples of solvable Lie algebras. In this case the geometry of structural uniformity can be described with the help of the foliation of the manifold $\mathfrak{B}$ by the family of surfaces. This foliation can be defined in a manner similar to the way in which it is applied in the geometric theory of Newtonian gravitation (e.g., Dixon, 1975). The surfaces of this foliation will then be the supports of dislocations of two-dimensional lattices [of rotation type, case $e(2)$; or of Lorentz rotation type, case $e(1, 1)$], combined with the slips along these surfaces. Linear defects of this type are observed in crystalline materials with clearly marked lamellar structure (e.g., graphite or liquid crystals of the smectic B type). Then the slip starts up easily in the planes parallel to the layers and is almost impossible in the planes that are not layers. Because of this, the location of dislocations (and also their Burgers vectors) is limited here to the planes that are layers (Hull and Bacon, 1984).

Let us return to the case of dislocations of rotation type. We will call such dislocations *disclinations*. This definition results from our supposition that, in the infinitesimal version, line defects of the nontranslational type are rather a type of distribution of dislocations than a separate kind of line defect [Part I, commentary after formula (96)]. It should be stressed that the above definition of (continuously distributed) disclinations is not generally accepted in the literature (e.g., de Wit, 1973; Minagava, 1979).

4. INFINITESIMAL MOTIONS

The discussion of properties of the closed teleparallelism (Section 3) revealed the basic role of infinitesimal g-translations in the description of the influence of dislocations on the material structure of the crystalline body. The question arises: what can be said about a structurally uniform body if we consider its infinitesimal g-motions?

Let $(\mathfrak{B}_g, \Phi)$, $\mathfrak{B}_g = (\mathfrak{B}, g)$, be a structurally uniform crystalline body. If $\underline{u}$ is a vector field on $\mathfrak{B}_g$, then we denote

$$l_g^2(\underline{u}) = g_{AB}u^A u^B \tag{33}$$

The field $\underline{u}$ is called an *infinitesimal g-motion* if

$$\delta l_g(\underline{u}) = 0 \tag{34}$$

The condition (34) is equivalent to the condition that $\underline{u}$ is the Killing vector field of the metric g:

$$\nabla_A u_B + \nabla_B u_A = 0, \qquad u_A = g_{AB}u^B \tag{35}$$

In equation (35) and also in other formulas in this section, $\nabla = \nabla^g$, i.e., ∇ denotes the covariant derivative based on the Levi-Civita connection for the metric g. In continuum mechanics, the infinitesimal motions corresponding to the right Cauchy-Green tensor [Part I, formula (71)] are called *virtual displacements* (Drobot, 1971).

It can be easily observed that the condition (34) is a generalization of the condition defining a rigid system of points M in the point Euclidean space E, i.e., a condition of the form (Drobot, 1971)

$$\forall P, Q \in M \qquad \delta(\mathbf{PQ} \cdot \mathbf{PQ}) = 0 \tag{36}$$

where $\mathbf{PQ}$ is the vector joining points P and Q, and the symbol δ denotes a variation that is a derivation, i.e., linear operation satisfying the Leibniz rule with respect to the scalar product $\underset{\sim}{a} \cdot \underset{\sim}{b}$ in the associated vector Euclidean space $\mathbf{E}$ (Part I, Appendix). The condition (36) is equivalent to the following two conditions [so-called *Poisson theorem* (Drobnot, 1971)]:

(i) For two arbitrary points $P, Q \in M$ there exists a vector $\underset{\sim}{\omega}$ such that

$$\delta \mathbf{PQ} - \underset{\sim}{\omega} \times \mathbf{PQ} = \underset{\sim}{0} \tag{37}$$

(ii) The vector $\underset{\sim}{\omega}$ is constant on M, i.e., independent of the choice of points $P, Q \in M$.

The condition (35) allows us to construct the infinitesimal counterpart of the vector $\underset{\sim}{\omega}$ through the introduction of the antisymmetric tensor ω_{AB} and vector ω^A dual to it, by

$$\omega_{AB} = \nabla_A u_B, \qquad \omega^A = \tfrac{1}{2} e^{ABC} \omega_{BC} \tag{38}$$

where e^{ABC} is the basic trivector of Riemannian manifold $\mathfrak{B}_g$ [Part I, formula (65) with the change of $\underset{\sim}{C}$ for $\underset{\sim}{g}$]. Then the condition (35) can be written in a form analogous to condition (37) (Drobot, 1971):

$$\nabla_A u_B - e_{ABC} \omega^C = 0 \tag{39}$$

However, the infinitesimal counterpart of condition (ii) from the Poisson theorem

$$\nabla_A \omega^B = 0 \tag{40}$$

is not generally fulfilled. In order to state this, let us consider the adjoint curvature tensor $\underset{\sim}{R}[\phi]$. From condition (35) it follows that (Nishioka, 1985)

$$R[\Phi]_{BCD}{}^A u_A = \nabla_D(\nabla_B u_C) \tag{41}$$

From equations (39) and (41) and from the ∇-parallelism of the basic trivector e^{ABC} it follows that the infinitesimal g-motion should satisfy the condition

$$R[\Phi]_{BCD}{}^A u_A = e_{BCA} \nabla_D \omega^A \tag{42}$$

Taking into consideration the fact that from (21) and (23) the representations

$$S_{AB}^{C}=\tfrac{1}{2}e_{ADB}\alpha^{DC}, \qquad R[\Phi]_{ABC}{}^{D}=\tfrac{1}{4}e_{ABE}e_{CFH}\alpha^{EF}\alpha^{HD} \tag{43}$$

follow, and denoting

$$\kappa^{A}{}_{B}=\nabla_{B}\omega^{A} \tag{44}$$

we can write the condition (42) in the form

$$\kappa^{A}{}_{B}=\tfrac{1}{4}e_{BCD}\alpha^{AC}\alpha^{DE}u_{E} \tag{45}$$

From the asymmetry of the adjoint curvature tensor in the first two indices it follows that

$$\kappa_{A}^{A}=0 \tag{46}$$

The trace-free tensor $\underline{\kappa}$ defined by (44) will be called a *curvature-twist* tensor. This tensor is an infinitesimal counterpart of the tensor of the same name considered in the continuous theory of disclinations (de Witt, 1970).

Let us denote by θ^{AB} the Einstein tensor corresponding to the adjoint curvature tensor, i.e. [Part I, formula (67)]

$$\begin{aligned} \theta^{AB}&=\tfrac{1}{4}e^{ACD}e^{BEF}R[\Phi]_{CDEF} \\ R[\Phi]_{ABCD}&=e_{ABE}e_{CDF}\theta^{EF} \\ R[\Phi]_{ABCD}&=g_{DE}R[\Phi]_{ABC}{}^{E} \end{aligned} \tag{47}$$

From (43) and (47) we obtain

$$\begin{aligned} \theta^{AB}&=\tfrac{1}{8}e^{EFB}e_{ECD}\alpha^{AC}\alpha^{D}{}_{F} \\ &=\tfrac{1}{8}(\alpha^{AC}\alpha^{B}{}_{C}-\alpha^{AB}\alpha^{D}{}_{D}), \qquad \alpha^{D}{}_{F}=\alpha^{DC}g_{CF} \end{aligned} \tag{48}$$

From formulas (21) and (48) it follows that in the case of lack of dislocations, which means that lattice lines of the nondeformed crystal are straight lines, the tensor θ^{AB} vanishes. On the other hand, vanishing of this tensor is equivalent to the vanishing of the adjoint curvature tensor (Part I, Section 4). So, we can treat the tensor θ^{AB} as the measure of bending of lattice metric lines and we will call it the *bending tensor* of these lines.

The curvature-twist tensor $\kappa^{A}{}_{B}$ and the bending tensor α^{AB} are connected by the following relation:

$$\kappa^{A}{}_{B}=e_{BCD}\theta^{AD}u^{C} \tag{49}$$

where $\underline{u}$ is a Killing vector field and $\kappa^{A}{}_{B}$ is defined by (38) and (44). The tensor of dislocation density α^{AB} [formula (21)] should satisfy the equation

$$\partial_{A}\alpha^{A}{}_{B}=0 \tag{50}$$

which is an identity following from the vanishing of the curvature tensor of the teleparallelism connection (Minagava, 1979). Equation (50) has the form of a continuity equation and can be interpreted as stating that the dislocations cannot end in the interior of the crystal (de Witt, 197).

Let us consider the condition (34) as defining a deformable continuum (with or without defects of the material structure). In physical applications we are often interested in the problem of finding all solutions of equation (35) fulfilling certain additional conditions. For example, Theorem 2 of Section 2 gives the necessary and sufficient conditions for the considered metric g to describe the metric structure of a structurally uniform crystalline body, and at the same time describes the solutions of (35) in the form $\underline{u} = \underline{E}_a$, $a = 1, 2, 3$. In this case the metric g can be interpreted as describing a distortion of the lattice that has no influence on the local metric properties of the body crystal structure [cf. Part I, commentary after condition (34)]. If the distortion of the lattice has no influence on the global metric properties of the body crystal structure, then, additionally, the metric g should be flat. Then the teleparallelism is closed (Wolf, 1972) and the solutions $\underline{u} = \underline{E}_a$, $a = 1, 2, 3$, of equation (35) have the form defined by formulas (3) and (9); it corresponds to a self-arrangement of dislocations that does not cause long-range stresses (Bilby *et al.*, 1958). From (47) it follows that in this case the bending tensor of metric lattice lines vanishes, though there are dislocations in the body; thus, the curvature-twist tensor also vanishes [relation (49)].

It is known that there exist such superficial arrangements of dislocations that manifest themselves in the bending of the crystal lattice, which is connected with the change of relative orientation between neighboring parts of the crystal. Such distortion of the crystal structure is not accompanied (in the absence of an external field) by macroscopic stress fields, but it is accompanied by macroscopic fields of so-called couple-stresses (Kröner, 1960). As a result, a crystal with dislocations reacts as a body with internal rotational degree of freedom [the so-called Cosserat continuum (Cosserat and Cosserat, 1909)]. It can be described by the imposition on the geometry of structural uniformity of an additional condition in the form of equation (40) or, what is equivalent, the condition [cf. Eq. (49)]

$$\theta^{A[D}u^{C]} = 0, \qquad \theta^{AB} \neq 0 \tag{51}$$

where θ^{AB} is given by (48). The couple-stresses will appear then as generalized reaction forces to these constraints (cf. Drobot, 1971). Such a material body can be called structurally uniform of *Cosserat type.*

5. THE GENERAL MATERIAL SPACE

The model of the structural uniformity of crystalline body is only the first approximation to real crystal structure. For example, we cannot describe complex lattices in this way. The influence of point defects on the lattice and on the line defects of this lattice is not considered in this model either.

On the other hand, existing geometrical theories of crystal lattice defects usually do not go beyond the notion of linear connection (e.g., Bilby, 1968). Therefore we should compare, first of all, the geometry of the structurally uniform body described by the consistent pair (Φ, g) with the geometry of the general linear connection.

Let Γ^A_{BC} be an arbitrary linear connection. Let us consider its decomposition in the form

$$\begin{aligned} \Gamma^A_{BC} &= \Lambda^A_{BC} - K_{BC}{}^A \\ K^{A}{}^{A}_{C} &= \underset{a}{e^A}\nabla_c \overset{a}{e}_B = -(\nabla_C \underset{a}{e^A}) \overset{a}{e}_B \end{aligned} \tag{52}$$

where ∇ is the covariant derivative for connection Γ^A_{BC}, Λ^A_{BC} is the connection determined by teleparallelism Φ having the smooth trivialization (2), and formulas (3) and (4) have been considered. Admission of the decomposition (52) does not limit the general character of the considerations, because if there is a defined teleparallelism Φ on the manifold, then each linear connection on it can be represented in this way (Eisenhart, 1972).

Let us introduce the following notations:

$$\nabla_{\underline{E}_b}{}^{E_a} = \omega_a{}^c{}_b \underline{E}_c, \qquad \nabla_B \underline{E}_b = \omega_B{}^a{}_b \underline{E}_a \tag{53}$$

Then

$$\omega_B{}^a{}_b = \overset{p}{e}_B \omega_p{}^a{}_b, \qquad \nabla_B \underset{a}{e^A} = \omega_a{}^b{}_c \underset{b}{e^A} \overset{c}{e}_B \tag{54}$$

and

$$\Gamma^A_{BC} = \underset{a}{e^A}(\partial_C \overset{a}{e}_B + \omega_B{}^a{}_b \overset{b}{e}_C) \tag{55}$$

The mixed-type components of the connection Γ^A_{BC} define the geometric object

$$\underset{\sim}{\omega}_B = \|\omega_B{}^a{}_b\|: \quad \mathfrak{B} \to gl(3), \qquad B = 1, 2, 3 \tag{56}$$

where $gl(3)$ is the Lie algebra of the Lie group $GL(R^3)$ of all nonsingular 3×3 matrices. This geometric object is called the *spinor connection* (Srivastava, 1983). For example, if

$$\underset{\sim}{\omega}_B = \overset{p}{e}_B \underset{\sim}{\gamma}_p, \qquad \underset{\sim}{\gamma}_p = \|\tfrac{1}{2}\gamma^c_{pb}\| \tag{57}$$

where the γ^c_{ab} are defined by (17), then the spinor connection defines the connection Γ of the form (14), i.e., the Levi-Civita connection for the metric g consistent with the teleparallelism Φ.

Let g be the metric tensor of a metric g consistent with Φ [formula (7) and conditions (15), (16)]. The metric g is invariant with respect to the following transformation of the vector basis $(\underline{E}_a)$:

$$\underline{E}'_a = \underline{E}_b L^b{}_a, \qquad E'^a = L_b{}^a E^b$$
$$\underline{L} = \|L^a{}_b(X)\| \in O_g(R^3), \qquad L_b{}^a = (\underline{L}^{-1})^a{}_b \tag{58}$$

i.e., with respect to the coordinate transformation

$$\underset{a}{e'}^A = \underset{b}{e}^A L^b{}_a, \qquad \overset{a}{e'}_A = L_b{}^a \overset{b}{e}_A \tag{59}$$

where $O_g(R^3)$ is the group conjugate in $GL(R^3)$ with $O(R^3)$, defined by representing the tensor g in the basis $\underline{E}_a$ [formula (7) and Part I, formula (8)]. This means that the comparison of the geometry described by the general connection Γ with the geometry of structural uniformity requires taking into account the fact that two groups of transformations act on the manifold $\mathfrak{B}$: the group of general transformations of all coordinates

$$X^A = f^A(X^B) \tag{60}$$

and the matrix group $O_g(R^3)$ acting nonhomogeneously as the coordinate transformation group of the (anholonomic) moving frame $(\underline{E}_a)$:

$$\underset{a}{e'}A(X) = \underset{b}{e}^A(X) L^b{}_a(X)$$
$$\|L^b{}_a(X)\| \in O_g(R^3), \qquad X = (X^A) \tag{61}$$

Geometric objects defined on such a space have "holonomic" indices $A, B, C, \ldots$ and "anholonomic" indices $a, b, c, \ldots$, e.g., T^{aA}, T^{aA}_{bB}, etc., corresponding to the rule of simultaneous coordinate transformations of these groups.

The spinor connection induces, by means of the representation (55), the covariant differentiation in this space

$$\bar{\nabla}_A = \partial_A + \Gamma_A(X) \tag{62}$$

acting in accordance with the rule (Rumer, 1956; Srivastava, 1983)

$$\bar{\nabla}_C T^{aA}_{bB} = \partial_C T^{aA}_{bB} + \Gamma_C(X) T^{aA}_{bB}$$
$$\Gamma_C(X) T^{aA}_{bB} = \omega_C a_p T^{pA}_{bB} - \omega_C{}^p{}_b T^{aA}_{pB} + \Gamma^A_{CD} T^{aD}_{bD} - \Gamma^D_{CB} T^{aA}_{bD} \tag{63}$$

We assume that this covariant differentiation satisfies the basic condition of the geometric description of the crystal structure: the condition of covariant constancy of the distribution (1) of lattice bases [cf. Eq. (12)]. This means that

$$\bar{\nabla}_B \overset{a}{e}_A = 0, \qquad \bar{\nabla}_B \underset{a}{e}^A = 0 \tag{64}$$

Such a geometric space can be called the *general material space* for a crystalline body with lattice defects.

From (62)-(64) it follows that

$$\bar{\nabla}_C T^{\dots B}_{A\dots} = \nabla_C T^{\dots B}_{A\dots} = \overset{a}{e}_A \dots \underset{b}{e}^B D_C T^{\dots b}_{a\dots}, \qquad T^{\dots B}_{A\dots} = \overset{a}{e}_{A\dots} \underset{b}{e}^B T^{\dots b}_{a\dots} \tag{65}$$

where

$$\bar{\nabla}_C T^{\dots b}_{a\dots} = D_C T^{\dots b}_{a\dots} = \partial_C T^{\dots b}_{a\dots} + \omega_C{}^b{}_p T^{\dots p}_{a\dots} - \omega_C{}^p{}_a T^{\dots a}_{p\dots} \tag{66}$$

In particular, for the matrix function

$$\underset{=}{A} = \|A^a{}_b\|: \quad \mathfrak{B} \to gl(3) \tag{67}$$

the formula (66) can be written in the form

$$\bar{\nabla}_C \underset{=}{A} = D_C \underset{=}{A} = \partial_C \underset{=}{A} + [\underset{=}{\omega}, \underset{=}{A}] \tag{68}$$

and the formula (65) in the form

$$\begin{gathered} \bar{\nabla}_C \circ Ad_{\underset{=}{e}} = Ad_{\underset{=}{e}} \circ D_C \\ Ad_{\underset{=}{e}}(\underset{=}{A}) = \underset{=}{e}\underset{=}{A}\underset{=}{e}^{-1}, \qquad \underset{=}{e} = \|\underset{a}{e}^A(X)\| \in GL(R^3) \end{gathered} \tag{69}$$

which is the relation characteristic for the gauge theory; but generally the differentiation D_C is not related to this theory.

The curvature of the spinor connection is described by differential operators P_{AB} defined by

$$P_{AB} = [D_A, D_B] \tag{70}$$

These operators act on the functions (67) according to the rule

$$P_{AB}(\underset{=}{A}) = [\underset{=}{P}_{AB}(\omega), \underset{=}{A}] \tag{71}$$

where

$$\underset{=}{P}_{AB}(\omega) = \partial_A \underset{=}{\omega}_B - \partial_B \underset{=}{\omega}_A + [\underset{=}{\omega}_A, \underset{=}{\omega}_B] \tag{72}$$

Because the decomposition (55) can be written in the form of the transformation

$$\sigma_{\underset{=}{e}}(\underset{=}{\omega}_A) = \underset{=}{e}\partial_A \underset{=}{e}^{-1} + \underset{=}{e}\underset{=}{\omega}_A \underset{=}{e}^{-1} \tag{73}$$

Therefore, for an arbitrary mapping

$$\underset{=}{L} = \|L^a{}_b\|: \quad \mathfrak{B} \to G \subset GL(R^3) \tag{74}$$

where G is a certain matrix Lie group, the spinor connection transforms according to the rule

$$\sigma_{\underset{=}{e}\underset{=}{L}}(\underset{=}{\omega}_A) = \underset{=}{L}\partial_A \underset{=}{L}^{-1} + \underset{=}{L}\sigma_{\underset{=}{e}}(\underset{=}{\omega}_A)\underset{=}{L}^{-1}, \qquad \underset{=}{e}\underset{=}{L} = \|\underset{a}{e}^A L^a{}_b\| \tag{75}$$

In particular, the rule (75) holds in the case when $G = O_g(R^3)$ [see transformations (58) and (59)]. Hence, and from the formulas (67)-(69), it follows that $\underline{P}_{AB}(\omega)$ also transforms according to the rule of gauge transformation (Srivastava, 1983):

$$\underline{P}_{AB}(\omega) \to Ad_{\underline{L}}(\underline{P}_{AB}(\omega)) = \underline{L}\underline{P}_{AB}(\omega)\underline{L}^{-1} \tag{76}$$

The indices in the object

$$R^a{}_{bCD}(\omega) = (\underline{P}_{CD}(\omega))^a{}_b \tag{77}$$

are all tensorial and

$$R_{BCD}{}^A(\Gamma) = \overset{b}{e}_B \underset{a}{e}{}^A R^A{}_{bCD}(\omega) \tag{78}$$

where $R_{BCD}{}^A(\Gamma)$ is the curvature tensor of the connection Γ^A_{BC}. So, we can call $\underline{P}_{AB}(\omega)$ the *spinor curvature tensor* (Srivastava, 1983). But it should be taken into account that here the Bianchi identity

$$D_A\underline{P}_{BC}(\omega) + D_C\underline{P}_{AB}(\omega) + D_B\underline{P}_{CA}(\omega) = 0 \tag{79}$$

is not generally fulfilled.

6. POINT DEFECTS AND GEOMETRICAL INTERACTIONS

Let (Φ, g) be a consistent pair and Γ an arbitrary linear connection. Let us denote

$$Q_{CAB} = \nabla_C g_{AB} \tag{80}$$

where ∇ is the covariant derivative for the connection Γ. From (52), (55), (65), and (66) it follows that

$$Q_{CAB} = 2K_{C(AB)} = \overset{a}{e}_A \overset{b}{e}_B D_C g_{ab}, \qquad K_{CAB} = K_{CA}{}^D g_{DB} \tag{81}$$

and

$$\omega_{C(ab)} = -\tfrac{1}{2} D_C g_{ab}, \qquad \omega_{Cab} = \omega_C{}^p{}_b g_{pa} \tag{82}$$

Let us denote

$$R_{ABCD}(\Gamma, g) = g_{DE} R_{ABC}{}^E \tag{83}$$

The so-called third identity for the curvature tensor $R_{ABC}{}^D(\Gamma)$ of the connection Γ has the form (Schouten, 1954)

$$R_{AB(CD)}(\Gamma, g) = -\nabla_{[A} Q_{B]CD} - S_{AB}{}^E(\Gamma) Q_{ECD} \tag{84}$$

where $S_{AB}{}^C(\Gamma)$ is the torsion tensor of the connection Γ and [from (52)]

$$S_{AB}{}^C(\Gamma) = \Gamma^C_{[AB]} - S_{AB}{}^C(\Phi) - K_{[AB]}{}^C, \qquad S_{AB}{}^C = \Lambda_{[AB]}{}^C \tag{85}$$

The condition

$$R_{AB(CD)}(\Gamma, g) = 0 \tag{86}$$

is equivalent to the condition that tensor Q_{CAB} in the form (Günther and Żórzwski, 1985)

$$Q_{CAB} = -2\nabla_C E_{AB}, \qquad E_{AB} = E_{BA} \tag{87}$$

which, by (80), can be written in the form

$$\nabla_C {}^G{}_{AB} = 0, \qquad G_{AB} = g_{AB} + 2E_{AB} \tag{88}$$

From (65), (66), (81), (82), (87), and (88) it follows that

$$\omega_{C(ab)} = D_C E_{ab} \tag{89}$$

and

$$D_C G_{ab} = 0, \qquad G_{ab} = g_{ab} + 2E_{ab} \tag{90}$$

where

$$\begin{gathered} E_{AB} = \overset{a}{e}_A \overset{b}{e}_B E_{ab} \\ G_{AB} = \overset{a}{e}_A \overset{b}{e}_B G_{ab}, \qquad g_{AB} = \overset{a}{e}_A \overset{b}{e}_B g_{ab} \end{gathered} \tag{91}$$

If G_{AB} is additionally a metric tensor, the connection Γ is the Cartan connection for G_{AB} and can be represented in the form (Schouten, 1954)

$$\Gamma^A_{BC} = \Gamma^A_{BC}(G) + S_{BC}{}^A(\Gamma) + 2S^A{}_{(BC)}(\Gamma, G) \tag{92}$$

where

$$S^A{}_{BC}(\Gamma, G) = G^{DA} G_{EC} S_{DB}{}^E(\Gamma) \tag{93}$$

and $\Gamma^A_{BC}(G)$ is the Levi-Civita connection for G_{AB}.

Let us consider the metric tensor G_{AB} defined in the following way. If $\underline{C}(X, t)$ is a right Cauchy-Green tensor induced on the body by its motion [Part I, formulas (70) and (71)] and

$$\underline{C}(X, t) = C_{ab}(X, t) E^a(X) \otimes E^b(X) \tag{94}$$

then

$$\begin{gathered} G_{ab} = G_{ab}(X, t) = L_a{}^c(X) L_b{}^d(X) C_{cd}(X, t) \\ \underline{L}(X) = \|L^a{}_b(X)\| \in GL^+(R^3) \end{gathered} \tag{95}$$

and

$$\begin{aligned} \underline{G} = \underline{G}(X, t) &= G_{ab}(X, t) E^a(X) \otimes E^b(X) \\ &= C_{ab}(X, t) E'^a(X) \otimes E'^b(X) \end{aligned} \tag{96}$$

where the basis $(\underline{E}'_a)$ is defined by (58) and (59) with the change of $O_g(R^3)$ for $GL^+(R^3)$. If

$$S_{AV}{}^C(\Phi) = 0 \tag{97}$$

then the vectors $\underset{\sim}{E}_a$, $a = 1, 2, 3$, create a natural basis of a certain curvilinear coordinate system on $\mathfrak{B}$ (Part I, Section 4) and in the case when $\underset{\sim}{L}$ is a field of nonorthogonal matrices such that

$$R_{ABC}{}^{D}(\underset{\sim}{G}(X, t)) \neq 0 \tag{98}$$

the metric tensor G_{AB} describes, according to Kröner's approach (Kröner, 1985; Günther and Żórawski, 1985), the influence of point defects on the elastically deformed crystalline body. Let us observe that the transformation (58) [with the change of $O_g(R^3)$ for $GL^+(R^3)$] can be considered as describing a certain teleparallelism Φ'. With that transformation, the metric tensor $\underset{\sim}{g}$ consistent with Φ passes (according to the postulate of metric uniformity; Part I, Section 3) to the metric tensor $\underset{\sim}{g}'$ of the form

$$\begin{gathered} \underset{\sim}{g}'(X) = g_{ab} E'^{a}(X) \otimes E'^{b}(X) = g'_{ab}(X) E^{a}(X) \otimes E^{b}(X) \\ g'_{ab}(X) = L_a{}^{c}(X) L_b{}^{d}(X) g_{cd} \end{gathered} \tag{99}$$

which is Φ'-parallel, but in general inconsistent with Φ'. Up to now, the case when only dislocations occur in the body has been described by a loosely defined teleparallelism, independent of the metric structure of the distorted lattice. With such an approach teleparallelism the Φ' appearing in Kröner's description of point defects does not have a univocal physical interpretation. In the proposed approach based on the application of the consistent pair (Φ, g) this ambiguity is eliminated.

We see that the tensor $K_{C(AB)}$ of the form [cf. (81) and (90)]

$$K_{C(AB)} = -\overset{a}{e}_A \overset{b}{e}_B D_C E_{ab} \tag{100}$$

can be considered as describing the influence of point defects and the elastic field on the geometry of structural uniformity of the crystalline body. The spinor connection related to this case has the form [cf. (82)]

$$\omega_A{}^{a}{}_{b} = \pi_A{}^{a}{}_{b} + g^{ap} D_A E_{pb}, \qquad \pi_A{}^{a}{}_{b} = g^{ap} \omega_{A[pb]} \tag{101}$$

If we do not consider interactions between dislocations, point defects, and the elastic field, then in the case

$$E_{ab} = 0 \tag{102}$$

it follows from (57) that

$$\pi_A{}^{a}{}_{b} = \tfrac{1}{2} \overset{p}{e}_A \gamma^{a}_{pb} \tag{103}$$

and from (81), (87), (91), and (102) it follows that

$$Q_{CAB} = 2K_{C(AB)} = 0 \tag{104}$$

Hence, and from (81) and (82), it follows that the considered dislocations are in reality the disclinations

$$\pi_A = \| \pi_A{}^a{}_b \|: \quad \mathfrak{B} \to o_g(3) \tag{105}$$

where $o_g(3)$ is the Lie algebra of the Lie group $O_g(R^3)$. In the case of the spinor connection of the form (101), (103) we have

$$S_{AB}{}^C(\Gamma) = \underset{a}{e}{}^C \overset{b}{e}_{[B} g^{pa} D_{A]} E_{pb} \tag{106}$$

with (Yano, 1955)

$$S_{AB}{}^C(\Phi) = -\tfrac{1}{2} \underset{a}{e}{}^C \overset{b}{e}{}_B \overset{p}{e}{}_A \gamma^a_{pb} \tag{107}$$

The above considerations suggest that if we want to take into account the existence of interactions between dislocations and point defects (and also the processes of creation and annihilation of defects), then we should consider the case when

$$R_{AB(CD)}(\Gamma, g) \neq 0 \tag{108}$$

Such interactions between the defects of the lattice are called *geometrical interactions* (Günther and Żórawski, 1985). The vanishing of geometrical interactions only in the case of disclinations can be treated as the justification for the distinction of this type of dislocation.

An example of a geometry permitting the occurrence of geometrical interactions, as well as their vanishing, is the Weyl–Cartan geometry [also called semimetric (Schouten, 1954)] characterized by the condition (81) and the condition

$$D_C g_{ab} = 2 g_{ab} \varphi_c \tag{109}$$

In this case

$$R_{ABCD}(\Gamma, g) = R_{AB(CD)}(\Gamma, g) = \partial_{[A} \varphi_{B]} g_{CD} \tag{110}$$

Geometrical interactions vanish if and only if

$$\varphi_C = \partial_C \varphi \tag{111}$$

If the condition (111) is fulfilled and

$$E_{ab} = \tfrac{1}{2}(e^{-2\varphi} - 1) g_{ab} \tag{112}$$

then

$$G_{AB} = e^{-2\varphi} g_{AB} \tag{113}$$

The formula (113) means that the influence of point defects on the crystal structure, with [condition (105)] or without [condition (97)] disclinations, takes the form of the dilatation field: $-2\varphi(X, t) > 1$ means that at the point $P \in \mathfrak{B}$ with the coordinates $X = X(P)$ (and at the instant t), the influence of interstitial atoms dominates; if $-2\varphi(X, t) < 1$, then the influence of nonoccupied sites in the lattice nodes (vacancies) dominates. This is the simplest model of the influence of point defects on the crystal structure.

7. CONCLUSIONS

The basic conclusion of this paper is that the pair (Φ, g) consisting of the teleparallelism Φ and the Riemannian metric g consistent with Φ is a geometric object realizing the following point of view: the lattice of the crystalline body with a continuous distribution of dislocations can be locally described as an ideal lattice in non-Euclidean space. This approach explains many problems that have been treated separately. For example, by using the consistent pair (Φ, g) we can explain the connection of disclinations (dislocations of rotation type) with the geometrical interactions (Section 6) as well as describe the reaction of the crystal with dislocations as a body with internal rotational degrees of freedom (Section 4).

The second important conclusion is that the transition from an ideal crystal to a crystal with (continuously distributed) dislocations can be described as a transition from a lattice basis consisting of Euclidean translation to a Bravais moving frame in the form of infinitesimal g-translations (Section 2). If the teleparallelism Φ is closed (Section 3), then these g-translations span a three-dimensional real Lie algebra and the types of continuous distributions of dislocations are described by the classification of such algebras (Section 3). Such a classification of (continuously distributed) dislocations is consistent with a topological classification of discrete linear defects of the lattice (Part I, Section 3).

In the proposed theory there also appear geometric objects so far not considered in the description of dislocations: the adjoint curvature tensor (Section 2) and connected with it the bending tensor of the metric lattice lines (Section 4). They allow one to understand the regularities of the dislocation distributions better than (as has been done up to now) with the use of only loosely defined teleparallelism (e.g., Section 4).

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